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MULTI-COMPONENT SYSTEMS AND STRUCTURES AND THEIR RELIABILITY

by

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Multi-component systems and structures and their reliability*)

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0. Summary.

A number of recent publications have dealt with problems of analyzing the performance of multi-component systems and evaluating their reliability. For example, a comprehensive theory of two-terminal networks was presented in [1] by Moore and Shannon who, among other results, have developed methods for obtaining highly reliable systems using components of low reliability; some of their procedures are credited to earlier work by von Neumann [2]. A discussion of complex systems interpreted as Boolean functions may be found in the paper [3] by Mine.

The present study deals with general classes of systems which contain two-terminal networks and most other kinds of systems considered previously as special cases, and investigates their combinatorial properties and their reliability. These classes consist, with several variants, of systems such that the more components perform the greater the probability that the system performs. For such systems it is shown that, if each component has reliability p and the reliability of the system is denoted by $h(p)$, then under mild additional assumptions $h(p)$ is an S-shaped function, i.e. its graph has the shape indicated in Fig. 3.2.4.1. Some of the consequences are these: there exists a critical value of p such that above that value the reliability of the system is greater than the reliability of a single component and below that value it is smaller; for p small the system has a reliability comparable to that of a series system, and for p large to that of a parallel system; by repeatedly iterating the system, i.e. by using replicas of the system instead of single components, one

obtains systems with reliability arbitrarily close to 1 if one starts with component reliability above the critical value, but with reliability arbitrarily close to 0 if one starts below that critical value.

1. Introduction

1.1. When a very complex device is constructed, consisting of a large number of components, it is often impossible to be quite sure that it will perform the task for which it was intended. Failures of components due to causes which are hard to anticipate and practically impossible to prevent may lead to failure of the entire structure. In such situations, the best one may strive for in designing the structure is to attain a high probability that it will perform its task. It has become customary to refer to the probability that a structure will perform the task for which it was designed as the reliability of that structure. A simplifying (and possibly not quite realistic) assumption is implied in this definition of reliability: it is assumed that a structure can only either perform or fail. To emphasize this assumption, we shall sometimes use the term dichotomic reliability for the probability that a structure will perform its task. It is possible to introduce and study a more general concept of reliability which accounts also for the possibility of partial performance. The present study, however, will be limited to dichotomic reliability.

1.2. A similar situation arises when, instead of a complex structure, one considers single mass-produced components. There again one can hardly expect to be sure that all components will perform and can only aim at a high probability that a component, when called upon, will perform. As before, the probability that a component will perform the function for which it is intended will be called the (dichotomic) reliability of that component.

1.3. It should be pointed out that the time element does not explicitly enter into our definition of dichotomic reliability. It may enter in what is meant by "performing" of "failing", e.g. when a structure is said to perform if it functions in a certain manner for at least 400 hours, and to fail if it breaks down before that time. But there also are practical situations where a structure is considered as performing when it is in working order for instantaneous use, and not necessarily for any length of time. Our dichotomic reliability includes the time-dependent reliability as a special case.

1.4. One of the main purposes of a mathematical theory of reliability is to develop means by which one can evaluate the reliability of a structure when the reliabilities of its components are known. The present study will be concerned with this kind of mathematical development. It will be necessary for this purpose to rephrase our intuitive concepts of structure, component, reliability etc. in more formal language, to restate carefully our assumptions, and to introduce an appropriate mathematical apparatus.

2. Formal description of a structure

2.1. In order to study the relationships between the reliabilities of the components of a structure and the reliability of that structure, one has to know how performance or failure of the various components affects performance or failure of the structure. For this purpose we shall describe the state of any device, single

component or complex structure, by a numerical value according to the code

$$(2.1.1)\qquad \begin{array}{lcl} \text{"performs"} & \longleftrightarrow & 1 \\ \text{"fails"} & \longleftrightarrow & 0 . \end{array}$$

If a structure consists of n components it will be called a <u>structure of order n</u>. The state of all components of such a structure will be described by the n-tuple of variables (a vector with n coordinates)

$$(2.1.2)\qquad (x_1, x_2, \ldots, x_n) = \underline{x}$$

where $x_i = 1$ means "i-th component performs" and $x_i = 0$ means "i-th component fails". All possible states of the n-tuple $\underline{x}$ are therefore given by the 2^n vectors $(0,0,\ldots,0)$, $(1,0,\ldots,0)$, $(0,1,\ldots,0)$, ... $(1,1,\ldots,1)$, that is by all the vertices of an n-dimensional unit cube. Some of these 2^n vectors will make our structure perform, and others will make it fail, so that the state of the structure may be written as a function of $\underline{x}$

$$(2.1.3)\qquad \phi(x_1, x_2, \ldots, x_n) = \phi(\underline{x})$$

which assumes the value 1 for those vectors $\underline{x}$ which make the structure perform and the value 0 for those which make it fail. The function $\phi(\underline{x})$ will be called the <u>structure-function</u> or, in short, <u>the structure</u>.

By analogy with terms used in the theory of circuits, any vector $\underline{x}$ for which $\phi(\underline{x}) = 1$ will be called a <u>path</u> for the

structure ϕ , and any vector $\underline{x}$ for which $\phi(\underline{x}) = 0$ a cut for that structure.

The number of components performing when the state of all components is given by a vector $\underline{x}$ is clearly given by the function of $\underline{x}$

$$S(\underline{x}) = \sum_{i=1}^{n} x_i$$

which will be called the size of $\underline{x}$.

The following abbreviations and definitions will be useful:

$\underline{0} = (0,0,\ldots,0)$ (state in which all components fail)

$\underline{1} = (1,1,\ldots,1)$ (state in which all components perform)

$\underline{x}' = \underline{1} - \underline{x}$

$\underline{x} \geq \underline{y}$ or $(x_1,x_2,\ldots,x_n) \geq (y_1,y_2,\ldots,y_n)$ when $x_i \geq y_i$ for $i = 1,2,\ldots,n$

$\underline{x} > \underline{y}$ when $x_i \geq y_i$ for $i = 1,2,\ldots,n$ and $x_j > y_j$ for at least one j.

2.2. Examples of structures.

2.2.1. Parallel Components.

A structure is said to consist of n components in parallel if it is so designed that it fails if and only if all components fail. This structure is described by

$$\phi(x_1,x_2,\ldots,x_n) = 1 - (1-x_1)(1-x_2)\ldots(1-x_n) = \text{Max}\,(x_1,\ldots,x_n)$$

and its only cut is $(0,0,\ldots,0) = \underline{0}$.

2.2.2. Components in series.

A structure which performs if and only if all its components perform is said to contain its components in series. It is described

by

$$\phi(x_1, x_2, \ldots, x_n) = x_1 \circ x_2 \circ \cdots \circ x_n = \min (x_1, x_2, \ldots, x_n) ,$$

and the only path for ϕ is $\underline{1}$.

2.2.3. "k out of n" structures.

Let

$$\phi(\underline{x}) = \begin{cases} 1 & \text{if } S(\underline{x}) \geq k \\ 0 & \text{if } S(\underline{x}) < k \end{cases}$$

where k is an integer, $0 \leq k \leq n$. This structure performs when at least k of its n components perform, and fails otherwise. We shall call it a "k out of n" structure. Clearly all $\underline{x}$ such that $S(\underline{x}) \geq k$ are paths, all others cuts. For $k = 1$ the k out of n structure reduces to the structure with parallel components, and for $k = n$ to the structure with components in series.

2.3. The dual structure.

2.3.1. For a given structure $\phi(\underline{x})$, we define the structure

$$\phi^D(\underline{x}) = 1 - \phi(\underline{1} - \underline{x}) = 1 - \phi(\underline{x}')$$

and call it the structure dual to ϕ . One verifies immediately that if a vector $\underline{x}$ is a path for ϕ then $\underline{x}'$ is a cut for ϕ^D and vice versa, and that $(\phi^D)^D = \phi$.

2.3.2. As an example consider the k out of n structure of 2.2.3. The dual structure is

$$\phi^D(\underline{x}) = \begin{cases} 1 & \text{if } S(\underline{x}) \geq n - k + 1 \\ 0 & \text{if } S(\underline{x}) < n - k + 1 , \end{cases}$$

so that for the k out of n structure the dual is the $n - k + 1$ out of n structure. In particular for $k = 1$ we find that the structure dual to that with parallel components is the series structure, and for $k = n$ we obtain the converse statement.

2.4. Path-numbers and cut-numbers.

A given structure ϕ may have paths and cuts of sizes ranging from 0 to n. We define the <u>path-numbers</u> for a given structure as

(2.4.1) A_j = number of paths for ϕ of size j, $j = 0,1,\ldots, n$.

and the <u>cut-numbers</u> for ϕ as

(2.4.2) A_j^* = number of cuts for ϕ of size j, $j = 0,1,\ldots,n$.

Let Υ_j be the set of all vectors of size j

(2.4.3) $$\Upsilon_j = \{\underline{x}: S(\underline{x}) = j\}.$$

Then clearly

(2.4.4) $$A_j = \sum_{\underline{x} \in \Upsilon_j} \phi(\underline{x}) , \quad A_j^* = \sum_{\underline{x} \in \Upsilon_j} [1 - \phi(\underline{x})]$$

and since Υ_j contains exactly $\binom{n}{j}$ vectors and each of them is either a path or a cut we have

(2.4.5) $$A_j + A_j^* = \binom{n}{j}.$$

Setting

A_j^D = number of paths for ϕ^D of size j

A_j^{D*} = number of cuts for ϕ^D of size j

we obtain

$$(2.4.6)\qquad A_j^D = A^*_{n-j}\ ,\quad A_j^{D*} = A_{n-j}\ .$$

2.5. Length and width of a structure.

By analogy with terms used in the theory of circuits we define for a structure ϕ

$$(2.5.1)\qquad \ell = \text{length of } \phi = \text{smallest } j \text{ such that } A_j > 0$$

$$(2.5.2)\qquad w = \text{width of } \phi = \text{smallest } k \text{ such that } A^*_{n-k} > 0\ .$$

Intuitively speaking, ℓ is the smallest number of components such that, if these components perform, the structure performs even if all other $n-\ell$ components fail. Similarly w is the smallest number of components such that if they fail the structure fails, even if all remaining $n-w$ components perform. For example, in a k out of n structure $\ell = k$, $w = n-k+1$.

According to (2.5.1) we have $A_j = 0$ for $j \leq \ell - 1$, hence by (2.4.5) $A^*_j = \binom{n}{j} > 0$ for $j \leq \ell - 1$. But according to (2.5.2) $A^*_{n-(w-1)} = 0$, hence $n-(w-1) \geq \ell$, and we obtain the inequality

$$(2.5.3)\qquad \ell + w \leq n+1\ .$$

2.6. Combination of structures.

Up to now we have always considered one structure of given order n. We shall now consider structures of different orders and in particular define an operation by which structures of a higher order can be obtained from structures of a lower order.

We shall say that the structure ϕ of order $n+1$ is a linear combination or, in short, a <u>combination</u> of the structures λ and

μ of order n , when the identity holds

$$(2.6.1)\quad \phi(x_1,x_2,\ldots,x_n,x_{n+1}) = x_{n+1}\lambda(x_1,x_2,\ldots,x_n) + (1 - x_{n+1})\,\mu(x_1,x_2,\ldots,x_n) .$$

Clearly if λ and μ are structures of order n then the right side of (2.6.1) is a structure of order $n+1$. Conversely, every structure ϕ or order $n+1$ can be represented as a combination of two structures of order n, i.e. can be written in the form (2.6.1), since one always has the identity

$$(2.6.2)\quad \phi(x_1,\ldots,x_n,x_{n+1}) = x_{n+1}\,\phi(x_1,\ldots,x_n,1)+(1-x_{n+1})\phi(x_1,\ldots,x_n,0).$$

This representation of a structure as a combination of structures of lower order can be carried further step by step, so that

$$\begin{aligned}
\phi(x_1,\ldots,x_{n-2},x_{n-1},x_n) &= x_n\phi(x_1,\ldots,x_{n-2},x_{n-1},1) + \\
&+ (1-x_n)\phi(x_1,\ldots,x_{n-1},0) = x_n[x_{n-1}\phi(x_1,\ldots,x_{n-2},1,1) + \\
&+ (1-x_{n-1})\phi(x_1,\ldots,x_{n-2},0,1)] + (1-x_n)[x_{n-1}\phi(x_1,\ldots,x_{n-2},1,0) + \\
&+ (1-x_{n-1})\phi(x_1,\ldots,x_{n-2},0,0)] = x_nx_{n-1}\,\phi(x_1,\ldots,x_{n-2},1,1) + \\
&+ x_n(1-x_{n-1})\phi(x_1,\ldots,x_{n-2},0,1) + (1-x_n)x_{n-1}\phi(x_1,\ldots,x_{n-2},1,0) + \\
&+ (1-x_n)(1-x_{n-1})\,\phi(x_1,\ldots,x_{n-2},0,0\,)
\end{aligned}$$

This procedure will terminate when one has reached structures of order 0, i.e. constants which are 0 or 1, and then it yields the representation which can be directly verified

(2.6.3) $\phi(x_1, x_2, \ldots, x_n) =$

$$= \sum_{\underline{y}} x_1^{y_1} x_2^{y_2} \ldots x_n^{y_n} (1-x_1)^{1-y_1} (1-x_2)^{1-y_2} \ldots (1-x_n)^{1-y_n} \phi(y_1, y_2, \ldots y_n)$$

where the sum is extended over all vectors $\underline{y}$ of order n.

2.7. Semi-coherent and coherent structures

2.7.1. Definitions.

Most structures occuring in practice are so designed that if a structure performs for a state $\underline{x}$ of its components then it performs for every state $\underline{y} > \underline{x}$, i.e. whenever some components which have value 0 (do not perform) in $\underline{x}$ are given value 1 (are made to perform). This leads us to the following definitions.

A structure ϕ is <u>semi-coherent</u> when

(2.7.1.1) $\phi(\underline{y}) \geq \phi(\underline{x})$ for all $\underline{y} > \underline{x}$.

A structure ϕ is <u>coherent</u> if it satisfies (2.7.1.1) and

(2.7.1.2) $\phi(\underline{0}) = 0, \quad \phi(\underline{1}) = 1$.

Thus a coherent structure is semi-coherent, with the additional property that it fails when all its components fail and performs when all components perform.

For given order n, there are only two structures which are semi-coherent but not coherent: the structure $\phi(\underline{x}) \equiv 0$ which fails for every state of its components, and the structure $\phi(\underline{x}) \equiv 1$ which performs for every state of its components. To see this one only has to note that $\underline{0} \leq \underline{x} \leq \underline{1}$ for every $\underline{x}$, hence by (2.7.1.1)

$$\phi(\underline{0}) \leq \phi(\underline{x}) \leq \phi(\underline{1}) \quad ,$$

and if ϕ is not coherent then either $\phi(\underline{0}) = 1$ hence $\phi(\underline{x}) = 1$ or $\phi(\underline{1}) = 0$ and $\phi(\underline{x}) = 0$.

If ϕ is semi-coherent (coherent) then ϕ^D is semi-coherent (coherent).

2.7.2. Theorem.

A structure ϕ of order $n+1$ is semi-coherent if and only if it can be represented as a linear combination of the form (2.6.1) with λ and μ a) semi-coherent and b) such that $\lambda(x_1,\ldots,x_n) \geq \mu(x_1,\ldots,x_n)$ for all $(x_1,\ldots,x_n)$. It is coherent if and only if, in addition to a) and b), either c$_1$) $\lambda(x_1,\ldots,x_n)=\mu(x_1,\ldots,x_n)$ for all $(x_1,\ldots,x_n)$ and $\lambda=\mu$ is a coherent structure, or c$_2$) $\lambda > \mu$ for some $(x_1,\ldots,x_n)$.

Proof:

If ϕ is semi-coherent then (2.6.2) is a representation of ϕ as a linear combination of $\lambda(x_1,\ldots,x_n) = \phi(x_1,\ldots,x_n,1) \geq$ $\geq \phi(x_1,\ldots,x_n,0) = \mu(x_1,\ldots,x_n)$, hence λ and μ are semi-coherent and $\lambda \geq \mu$. If ϕ is coherent then either $\lambda(x_1,\ldots,x_n) = \phi(x_1,\ldots,x_n,1) = \phi(x_1,\ldots,x_n,0) = \mu(x_1,\ldots,x_n)$ for all $(x_1,\ldots,x_n)$, and $\lambda(1,\ldots,1) = \mu(1,\ldots,1) = \phi(1,\ldots,1,1) = 1$, $\lambda(0,\ldots,0) = \mu(0,\ldots,0) = \phi(0,\ldots,0,0) = 0$ and λ and μ are coherent; or there is an $(x_1^*,\ldots,x_n^*)$ such that $\lambda(x_1^*,\ldots,x_n^*) >$ $> \mu(x_1,\ldots,x_n^*)$.

If ϕ can be written

$$\begin{aligned}\phi(x_1,\ldots,x_n,x_{n+1}) &= x_{n+1}\lambda(x_1,\ldots,x_n) + (1-x_{n+1})\mu(x_1,\ldots,x_n) = \\ &= x_{n+1}[\lambda(x_1,\ldots,x_n) - \mu(x_1,\ldots,x_n)] + \\ &+ \mu(x_1,\ldots,x_n)\end{aligned}$$

with λ and μ semi-coherent then

$$\Delta = \phi(y_1,\ldots,y_n,y_{n+1}) - \phi(x_1,\ldots,x_n,x_{n+1}) =$$

$$= (y_{n+1}-x_{n+1})\,[\lambda(y_1,\ldots,y_n) - \mu(y_1,\ldots,y_n)] +$$

$$+ x_{n+1}[\lambda(y_1,\ldots,y_n)-\lambda(x_1,\ldots,x_n)] + (1-x_{n+1})[\mu(y_1,\ldots,y_n)-\mu(x_1,\ldots,x_n)].$$

Assume $y_1 \geq x_1,\ldots,y_{n+1} \geq x_{n+1}$. If a) and b) are satisfied then either $x_{n+1} = 0$ and $\Delta = y_{n+1}[\lambda(y_1,\ldots,y_n) - \mu(y_1,\ldots,y_n)] + [\mu(y_1,\ldots,y_n) - \mu(x_1,\ldots,x_n)] \geq 0$, or $1 = x_{n+1} \leq y_{n+1} = 1$ and $\Delta = \lambda(y_1,\ldots,y_n) - \lambda(x_1,\ldots,x_n) \geq 0$, hence ϕ is semi-coherent. If in addition c$_1$) holds then $\phi(x_1,\ldots,x_n,x_{n+1}) = \lambda(x_1,\ldots,x_n)$ for all $x_1,\ldots,x_n$, x_{n+1} and ϕ is coherent. If a), b) and c$_2$) hold then there is a vector $x_1^*,\ldots,x_n^*$ such that $1 = \lambda(x_1^*,\ldots,x_n^*) > \mu(x_1^*,\ldots,x_n^*) = 0$, hence $\phi(1,\ldots,1,1) = 1$, $\phi(0,\ldots,0,0) = 0$ and ϕ is coherent.

2.7.3. The preceding theorem suggests the following constructive procedure for obtaining all semi-coherent structures.

Let $\mathcal{C}_n$, for $n=1,2,\ldots,$ denote the class of structures of order n defined as follows.

$\mathcal{C}_1$ consists of the three structures of order one:

$\beta_0(x_1)$ such that $\beta_0(0) = \beta_0(1) = 0$

$\beta_1(x_1)$ such that $\beta_1(0) = 0, \beta_1(1) = 1$

$\beta_2(x_1)$ such that $\beta_2(0) = \beta_2(1) = 1$.

In concrete terms, these are one-component structures such that β_0 never performs (circuit with one contact, grounded), β_1 performs if and only if the component performs (circuit with one operative contact), β_3 always performs (circuit with one contact, shorted).

We define $\mathcal{C}_2$ as the class of all structures of order 2 of the form

$$\phi(x_1,x_2) = x_2\lambda(x_1) + (1-x_2)\mu(x_1)$$

where $\lambda, \mu \in \mathcal{C}_1$ and $\lambda(x_1) \geq \mu(x_1)$ for $x_1 = 0,1$.

Having defined $\mathcal{C}_n$ we define recursively $\mathcal{C}_{n+1}$ as the class of all structures of order $n+1$ of the form

$$\phi(x_1,\ldots,x_n,x_{n+1}) = x_{n+1}\lambda(x_1,\ldots,x_n) + (1-x_{n+1})\mu(x_1,\ldots,x_n)$$

where $\lambda, \mu \in \mathcal{C}_n$ and $\lambda(x_1,\ldots,x_n) \geq \mu(x_1,\ldots,x_n)$ for all $(x_1,\ldots,x_n)$.

Clearly $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \ldots \subset \mathcal{C}_n \subset \mathcal{C}_{n+1} \subset \ldots$, and from 2.7.2 it follows that $\mathcal{C}_n$ is exactly the class of all semi-coherent structures of order n.

2.7.4. An inequality for path-numbers.

Let ϕ be a semi-coherent structure of order n and $A_0, A_1, \ldots, A_n$ its path-numbers. Then

$$(2.7.4.1) \qquad (j+1)\, A_{j+1} \geq (n-j)A_j \qquad \text{for } j=0,1,\ldots,n,$$

or equivalently,

$$(2.7.4.2) \qquad \frac{A_j}{\binom{n}{j}} \leq \frac{A_{j+1}}{\binom{n}{j+1}} \leq 1 .$$

Proof: for two consecutive integers $0 \leq j,\ j+1 \leq n$ we consider the sets $\mathscr{V}_j$, $\mathscr{V}_{j+1}$ of vectors of size j and $j+1$ and, for the purpose of this proof only, denote the elements of $\mathscr{V}_j$ by $\underline{x}$ and of $\mathscr{V}_{j+1}$ by $\underline{y}$. For each specific $\underline{x}$, we denote by $\mathscr{Y}(\underline{x})$ the set of all $\underline{y}$ such that $\underline{y} \geq \underline{x}$:

$$\mathscr{Y}(\underline{x}) = \{\underline{y}:\ \underline{y} \geq \underline{x}\}.$$

Each set $\mathscr{Y}(\underline{x})$ contains $(n-j)$ different $\underline{y}$'s. Since for each y there are $(j+1)$ different $\underline{x}$'s such that $\underline{y} \geq \underline{x}$, the collection of all elements of all sets $\mathscr{Y}(\underline{x})$ as $\underline{x}$ ranges through $\mathscr{V}_j$ forms $(j+1)$ replicas of $\mathscr{V}_{j+1}$. Therefore

$$\sum_{\underline{x} \in \mathscr{V}_j} \sum_{\underline{y} \in \mathscr{Y}(\underline{x})} \phi(\underline{y}) = (j+1) \sum_{\underline{y} \in \mathscr{V}_{j+1}} \phi(\underline{y}) = (j+1)\, A_{j+1}$$

and since

$$\sum_{\underline{x} \in \mathscr{V}_j} \sum_{\underline{y} \in \mathscr{Y}(\underline{x})} \phi(\underline{y}) \geq \sum_{\underline{x} \in \mathscr{V}_j} \sum_{\underline{y} \in \mathscr{Y}(\underline{x})} \phi(\underline{x}) = \sum_{\underline{x} \in \mathscr{V}_j} \phi(\underline{x})(n-j) = (n-j)\, A_j$$

we obtain (2.7.4.1).

2.7.5. Inequality (2.7.4.1) points to the fact that the requirement of a structure being semi-coherent imposes considerable restrictions on the path-numbers A_j. Several observations are worth making in this connection.

2.7.5.1. Inequality (2.7.4.1) is necessary but not sufficient for a structure being semi-coherent, as is shown by the following example. The structure of order 3

$\phi(0,0,0) = 0$, $\phi(0,0,1) = 1$, $\phi(0,1,0) = 1$
$\phi(0,1,1) = 1$, $\phi(1,0,0) = 0$, $\phi(1,0,1) = 1$
$\phi(1,1,0) = 0$, $\phi(1,1,1) = 1$

has the paths

of size 0: none
of size 1: $(0,0,1)$, $(0,1,0)$
of size 2: $(0,1,1)$, $(1,0,1)$
of size 3: $(1,1,1)$

hence its path-numbers are

$$A_0 = 0 , A_1 = 2 , A_2 = 2 , A_3 = 1$$

and (2.7.4.1) is fulfilled. But ϕ is not semi-coherent since e.g. $(1,1,0) > (0,1,0)$ and $\phi(1,1,0) = 0 < 1 = \phi(0,1,0)$.

2.7.5.2. If in the sequence of path-numbers $A_0, A_1, \ldots, A_n$ of a semi-coherent structure one has $A_j = 0$ then $A_i = 0$ for all $i \leq j$, and if one has $A_k = \binom{n}{k}$ then $A_i = \binom{n}{i}$ for all $i \geq k$. Both statements follow from (2.7.4.2) .

2.7.5.3. For ϕ coherent, obviously $A_0 = 0$, $A_n = 1$. Furthermore for the length and width of ϕ one has $\ell \geq 1$, $w \geq 1$, and inequality (2.5.3) becomes

(2.7.5.3) $\qquad 2 \leq \ell + w \leq n + 1$.

2.7.6 Reduction of coherent structures.

2.7.6.1 After a structure is designed, it may be possible to simplify it by omitting components which turn out to be inessential for its performance. To do this it would be helpful to have criteria by which one can tell whether a given structure has such inessential components. A criterium of this kind will be given in 2.7.6.4, but before it can be stated we need the following definitions.

2.7.6.2 Definitions

A vector $\underline{z}$ is a minimal path for the coherent structure $\phi(\underline{x})$, when $\phi(\underline{z}) = 1$ but for every $\underline{x} < \underline{z}$ we have $\phi(\underline{x}) = 0$.

A component x_i is inessential or a dummy component when

$$\phi(x_1,\ldots,x_{i-1}, 0, x_{i+1},\ldots,x_n) = \phi(x_1,\ldots,x_{i-1},1,x_{i+1},\ldots,x_n)$$

for all values of $x_1,\ldots, x_{i-1}, x_{i+1}, \ldots, x_n$. Otherwise x_i is an essential component.

A structure ϕ is called irreducible if all its components are essential.

2.7.6.3 Remarks.

One verifies easily that if $\underline{x}$ is a path for a coherent structure then there is at least one minimal path $\underline{z}$ such that $\underline{z} \leq \underline{x}$.

If a structure $\phi(x_1, \ldots, x_n)$ has a dummy component x_i, then we have identically $\phi(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) =$

$$= \phi(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = \psi(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$$

so that the component x_i can be omitted and the resulting structure ψ with only n-1 components is equivalent with the original structure ϕ.

Let us now consider a structure $\phi(x_1,\ldots,x_n)$, and assume that all its minimal paths are $\underline{z}^{(1)}, \underline{z}^{(2)}, \ldots, \underline{z}^{(k)}$.
Each of these minimal paths is a vector

$$\underline{z}^{(j)} = (z_1^{(j)}, z_2^{(j)}, \ldots, z_n^{(j)}), \quad j = 1, 2, \ldots, k,$$

with coordinates 0 or 1. We consider the vector defined by

$$(2.7.6.3) \quad \underline{z}^* = (\max_j z_1^{(j)}, \max_j z_2^{(j)}, \ldots, \max_j z_n^{(j)}) = (z_1^*, z_2^*, \ldots, z_n^*).$$

The meaning of this vector is this: if $z_i^* = 1$ then the coordinate x_i assumes the value 1 for at least one of the minimal paths, that is the i-th component is required to perform for at least one minimal path; if $z_i^* = 0$ then the i-th coordinate does not perform for any of the k minimal paths. In particular, if $\underline{z}^* = \underline{1}$ then everyone of the n components of the structure must perform for some minimal path.

2.7.6.4 Theorem.

Those coordinates of the vector $\underline{z}^*$ in (2.7.6.3) which have the value 1 correspond to essential components, and those which have the value 0 to dummy components. In particular, a necessary and sufficient condition for a coherent structure being irreducible is $\underline{z}^* = \underline{1}$.

Proof: without loss of generality we may consider the coordinate z_1^*. If $z_1^* = 1$ then there is a minimal path $\underline{z}^{(n)} = (z_1^{(n)}, z_2^{(n)}, \ldots, z_n^{(n)})$, say, such that $z_1^{(n)} = 1$ hence $\underline{z}^{(n)} = (1, z_2^{(n)}, \ldots, z_n^{(n)})$. Since it is minimal and $(0, z_2^{(n)}, \ldots, z_n^{(n)}) < (1, z_2^{(n)}, \ldots, z_n^{(n)})$, we have $\phi(1, z_2^{(n)}, \ldots, z_n^{(n)}) = 1$ and $\phi(0, z_2^{(n)}, \ldots, z_n^{(n)}) = 0$, and x_1 is an essential component.

If $z_1^* = 0$, then $x_1 = 0$ for all minimal paths. Consider any vector $\underline{x} = (x_1, x_2, \ldots, x_n)$; it is either a path or a cut. We shall show that in either case $\phi(0, x_2, \ldots, x_n) = \phi(1, x_2, \ldots, x_n)$, that is x_1 is a dummy component.

If $\underline{x}$ is a path then there exists a minimal path $\underline{z}^{(s)} = (z_1^{(s)}, z_2^{(s)}, \ldots, z_n^{(s)})$, say, such that $\underline{z}^{(s)} \leq \underline{x}$ and $\underline{z}^{(s)} = (0, z_2^{(s)}, \ldots, z_n^{(s)}) \leq (0, x_2, \ldots, x_n) \leq (x_1, x_2, \ldots, x_n) = \underline{x} \leq (1, x_2, \ldots, x_n)$, hence $\phi(\underline{z}^{(s)}) = 1 \leq \phi(0, x_2, \ldots, x_n) \leq \phi(1, x_2, \ldots, x_n)$ and $1 = \phi(0, x_2, \ldots, x_n) = \phi(1, x_2, \ldots, x_n)$.

Now consider an $\underline{x}$ which is a cut. Then either $x_1 = 0$ or $x_1 = 1$. If $x_1 = 0$, hence $\underline{x} = (0, x_2, \ldots, x_n)$ a cut, then $(1, x_2, \ldots, x_n)$ must also be a cut; for if $(1, x_2, \ldots, x_n)$ were a path then the argument of the preceding paragraph applied to this vector would yield $1 = \phi(0, x_2, \ldots, x_n) = \phi(1, x_2, \ldots, x_n)$ in contradiction with $(0, x_2, \ldots, x_n)$ being a cut. Hence if $x_1 = 0$ then $\phi(0, x_2, \ldots, x_n) = \phi(1, x_2, \ldots, x_n) = 0$. If $x_1 = 1$, hence $\underline{x} = (1, x_2, \ldots, x_n)$ a cut, then $0 = \phi(1, x_2, \ldots, x_n) \geq \phi(0, x_2, \ldots, x_n)$ and again $\phi(0, x_2, \ldots, x_n) = \phi(1, x_2, \ldots, x_n) = 0$, which completes the proof.

2.7.6.5 Remark.

Theorem 2.7.6.4 suggests the following procedure for simplifying a coherent structure ϕ: first one lists all minimal paths for ϕ; then one prepares a list of all those components which must perform (have value 1) in at least one minimal path. If this list contains all n components of ϕ then this structure is irreducible. If a component does not occur in this list then that component is inessential and can be omitted.

2.8 Composition of structures.

In practical designing of structures it often happens that

the design is prepared in stages, so that first a structure of order m is conceived, and then for some or all of its m components one substitutes other multi-component structures. This is for example the case when in planning electric circuits self-contained packages of components called "moduls" are used as "components" of a circuit. The formal description of this process of superimposing structures is introduced by the following definition.

Let $\Gamma(x_1,\ldots,x_n)$ be a structure of order n, and

$\Psi_1(y_1,\ldots,y_{k_1})$ a structure of order k_1

$\Psi_2(y_{k_1+1},\ldots,y_{k_1+k_2})$ a structure of order k_2

$\vdots$

$\Psi_n(y_{k_1+k_2+\ldots+k_{n-1}+1},\ldots,y_{k_1+k_2+\ldots+k_{n-1}+k_n})$

a structure of order k_n

Then the structure

$$\chi(y_1,y_2,\ldots,y_{k_1+k_2+\ldots+k_n}) = \Gamma(\Psi_1,\Psi_2,\ldots,\Psi_n)$$

is called the <u>composition of</u> $\Psi_1, \Psi_2,\ldots, \Psi_n$ <u>into</u> Γ.

The blocks of components $(y_1,\ldots,y_{k_1})$, $(y_{k_1+1},\ldots,y_{k_1+k_2})$, etc. may overlap. In particular, all of them may consist of the same m components $(y_1,\ldots,y_m)$, so that the resulting structure

$$\Gamma[\Psi_1(y_1,\ldots,y_m), \Psi_2(y_1,\ldots,y_m),\ldots, \Psi_n(y_1,\ldots,y_m)] = \chi(y_1,\ldots,y_m)$$

is of order m.

One verifies that if γ and $\psi_1, \ldots, \psi_n$ are all semi-coherent then χ is semi-coherent, and if $\gamma, \psi_1, \ldots, \psi_n$ are coherent then χ is coherent.

2.8.2. Combination of structures, as defined in 2.6, is a special case of composition, since (2.6.1) can be obtained by the composition of

$$u = x_{n+1}, \quad v = \lambda(x_1, \ldots, x_n), \quad w = \mu(x_1, \ldots, x_n)$$

into

$$(2.8.2.1) \quad \gamma(u, v, w) = uv + (1-u)w \quad .$$

One can therefore, beginning with the structure γ of order 3 given in (2.8.2.1) which is coherent, and using semi-coherent λ and μ of increasing orders, obtain exactly the same family of structures which can be obtained beginning with $\beta_0, \beta_1, \beta_2$ of 2.7.3 and proceeding by successive combinations.

3. The reliability function.

3.1. Definition of reliability function.

3.1.1. As indicated in section 1, one of the main aims of the mathematical theory of reliability is to evaluate the probability that a given structure will perform. Having previously discussed formal properties of structures, we shall now assume that components of a given structure may perform or fail in a random manner and derive various statements about the reliability of the structure.

3.1.2. Let ϕ be a structure of order n. We assume that its n components are independent random variables

$$\underline{X} = (X_1, X_2, \ldots, X_n) ,$$

each with the same probability distribution

$$(3.1.2.1) \quad Pr\{X_i = 1\} = p, \quad Pr\{X_i = 0\} = 1 - p = q ,$$

so that

$$\phi(\underline{X}) = \phi(X_1, \ldots, X_n)$$

is also a random variable capable of the values 1 and 0. We now define the <u>reliability function</u> of the structure ϕ as

$$(3.1.2.2) \quad h_\phi(p) = Pr\{\phi(\underline{X}) = 1\} = E[\phi(\underline{X})] ,$$

and devote the remainder of this paper to the study of reliability functions.

3.2. General properties of reliability functions.

3.2.1. From (3.1.2.1), (3.1.2.2) and (2.4.1) follows immediately

$$(3.2.1) \quad h_\phi(p) = \sum_{j=0}^{n} A_j p^j (1 - p)^{n-j} .$$

From this it is clear that $h_\phi(0) = 0$ if and only if $A_0 = 0$ and $h_\phi(1) = 1$ if and only if $A_n = 1$.

3.2.2. If the structure ϕ of order $n+1$ is a combination of structures λ and μ of order n , then by taking mathematical expectations on both sides of (2.6.1) and making use of the independence of $X_1, X_2, \ldots, X_{n+1}$ one obtains

(3.2.2) $\qquad h_{\phi}(p) = p h_{\lambda}(p) + (1 - p) h_{\mu}(p)$.

3.2.3. We now consider structures obtained by composition in the following manner. Let $\Phi(x_1, x_2, \ldots, x_n)$ be a structure of order n, $\Psi(y_1, y_2, \ldots, y_m)$ a structure of order m, and let $Y_1, Y_2, \ldots, Y_m, Y_{m+1}, Y_{m+2}, \ldots, Y_{2m}, \ldots, Y_{(n-1)m+1}, Y_{(n-1)m+2}, \ldots, Y_{nm}$ be independent random variables, all with the same probability distribution

$$\Pr\{Y_i = 1\} = p, \qquad \Pr\{Y_i = 0\} = 1 - p = q .$$

Then the structure of order nm defined by

$$\chi(\Psi) = \chi(Y_1, Y_2, \ldots, Y_{nm}) =$$

$$= \Phi[\Psi(Y_1, \ldots, Y_m), \Psi(Y_{m+1}, \ldots, Y_{2m}), \ldots, \Psi(Y_{(n-1)m+1}, \ldots, Y_{nm})]$$

has the reliability function

(3.2.3) $\qquad h_{\chi}(p) = h_{\Phi}(h_{\Psi}(p))$.

3.2.4. The reliability functions of most practical structures have the following plausible qualitative properties:

a) $h(0) = 0$ and $h(1) = 1$

b) $h'(p) > 0$ for $0 < p < 1$

c) $h(p) < p$ in some neighborhood of 0
$h(p) > p$ in some neighborhood of 1

and there is exactly one root of the equation $h(p) = p$ in the open interval $(0,1)$.

Necessary and sufficient conditions for ϕ to have h_ϕ with property a) were stated in 3.2.1. We shall also obtain conditions on a structure ϕ which are necessary and sufficient in order that h_ϕ have property b), or a more precisely defined property c). Before doing this, however, some intuitive comments on these properties may be in order.

Property b) tells that the reliability of the structure increases as the component reliability p increases, and is encountered for practically all structures.

When $h(p)$ has property c) we will say that $h(p)$ is <u>S-shaped</u>, since its graph has then the general form indicated in Fig. 3.2.4.1. A formal theory of S-shaped functions will be presented later, but already in this qualitative discussion we may indicate some consequences of $h(p)$ being S-shaped.

If $h(p)$ is S-shaped and equation $h(p) = p$ has the root $0 < \hat{p} < 1$, then we have $h(p) < p$ and the structure is less reliable than any single component as long as $p < \hat{p}$, and similarly $h(p) > p$ and the structure is more reliable than any single component when $p > \hat{p}$.

Furthermore, let $\phi(\underline{x})$ be a structure with an S-shaped reliability function $h(p)$, and let the sequence of <u>iterated compositions</u> of ϕ be defined as the sequence of structures

$$\begin{aligned}
\phi_1 &= \phi \\
\phi_2 &= \phi(\phi_1) \\
&\vdots \\
\phi_k &= \phi(\phi_{k-1}) \\
&\vdots
\end{aligned}$$

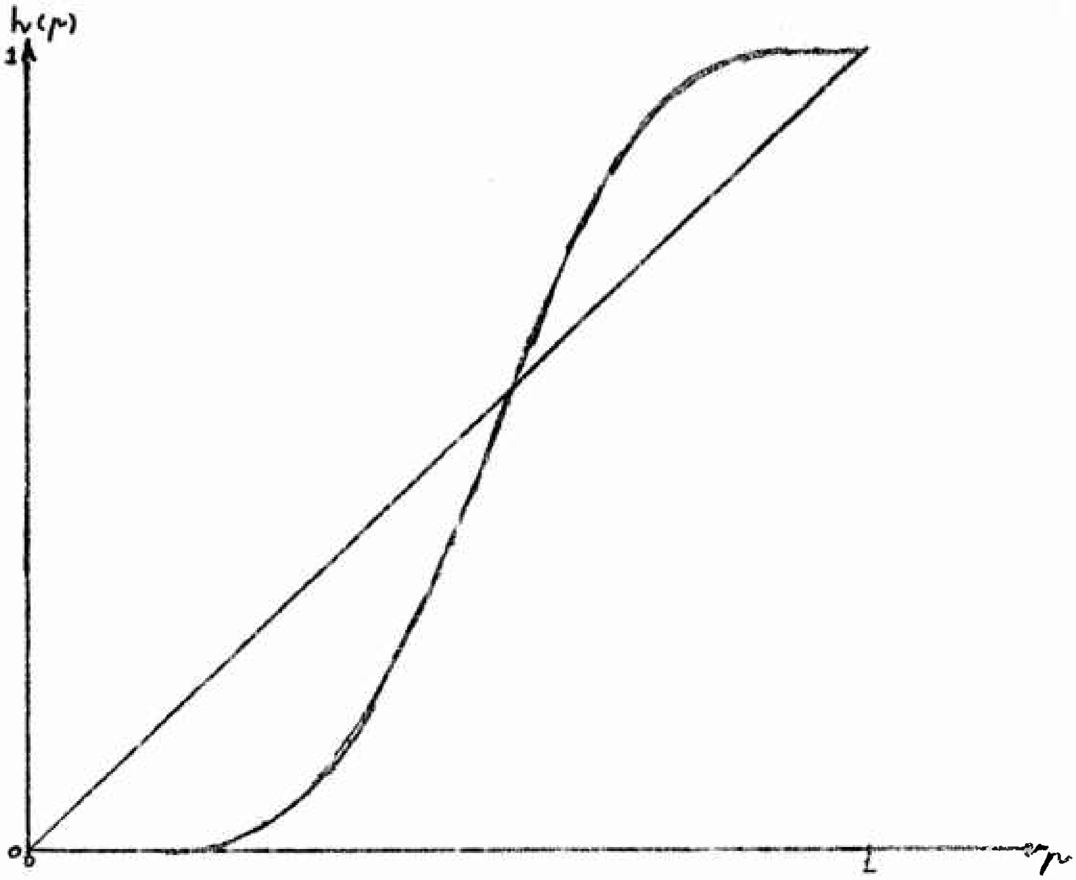

Fig. 3.2.4.1

S-shaped function

(reliability function for a 6 out of 10 structure)

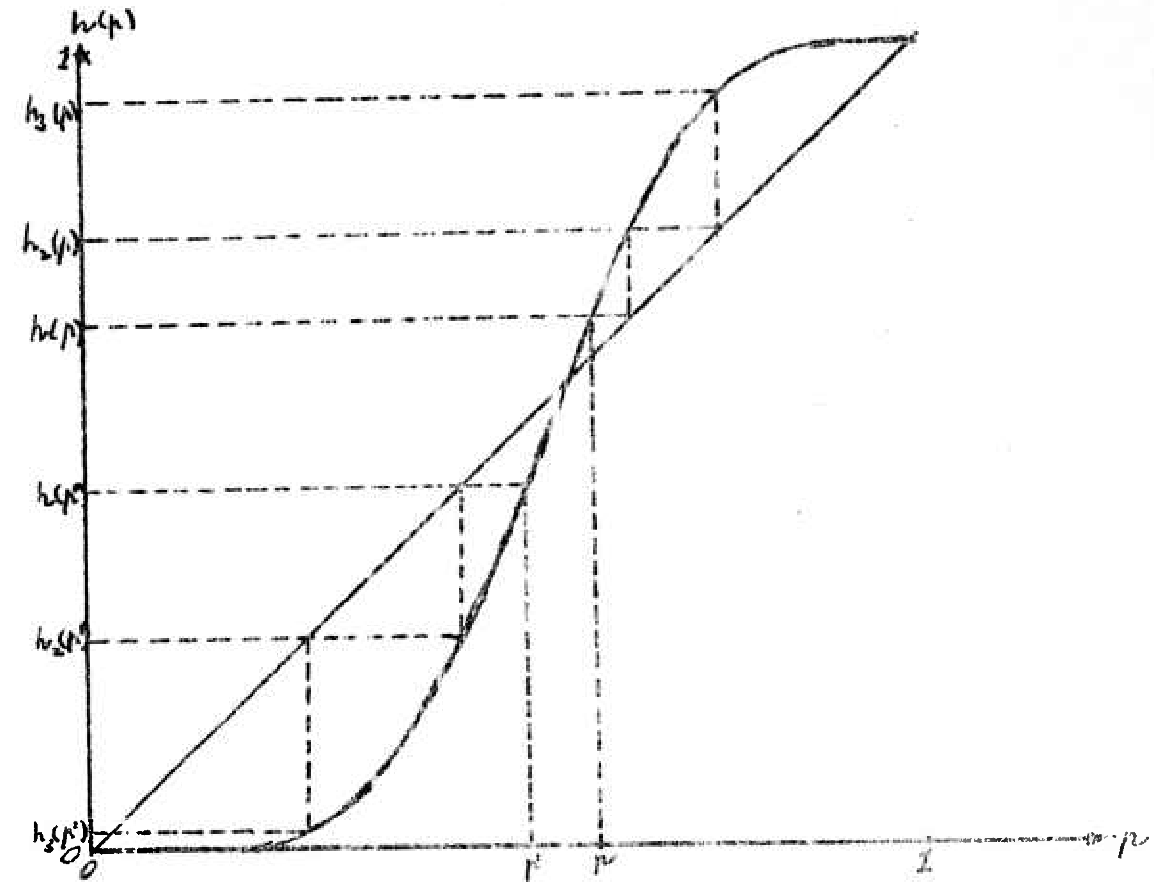

Fig. 3.2.4.2.

Iterated composition of reliability function for 6 out of 10 structure

According to 3.2.3. the corresponding reliability functions are

$$h_1(p) = h(p)$$
$$h_2(p) = h[h_1(p)]$$
$$\vdots \qquad \vdots$$
$$h_k(p) = h[h_{k-1}(p)]$$
$$\vdots \qquad \vdots$$

Let again $h(\hat{p}) = \hat{p}$, $0 < \hat{p} < 1$. Then it can be seen by an argument indicated by Figure 3.2.4.2. that

$$h_1(p) > h_2(p) > \ldots > h_k(p) \longrightarrow 0 \quad \text{if} \quad p < \hat{p} \ ,$$

$$h_1(p) < h_2(p) < \ldots < h_k(p) \longrightarrow 1 \quad \text{if} \quad p > \hat{p} \ ,$$

so that by iterated compositions of $\mathfrak{J}$ one can obtain structures with reliabilities as close to 1 as desired if the initial component reliability is $p > \hat{p}$, but tending to 0 if $p < \hat{p}$.

3.2.5. Example: k out of n structures.

It is easily seen that the structure described in 2.2.3 has the reliability function

(3.2.5.1) $$h(p;k,n) = \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i} \ .$$

For $n = 1,2,\ldots, 25$, the roots $\hat{p}_{k,n}$ of the equation

(3.2.5.2) $$h(p;k,n) = p \ , \qquad 0 < p < 1$$

may be read off from Table 3.2.5.

TABLE 3.2.5

Solutions $\hat{p}_{k,n}$ of the equation $h(p;k,n) = p$ for $p \in (0,1)$.

n \ k	2	3	4	5	6	7	8	9	10	11	12	13
3	.500											
4	.232											
5	.131	.500										
6	.083	.347										
7	.058	.256	.500									
8	.042	.197	.395									
9	.032	.158	.322	.500								
10	.025	.129	.268	.421								
11	.020	.108	.228	.361	.500							
12	.016	.092	.197	.314	.437							
13	.013	.080	.172	.276	.387	.500						
14	.011	.070	.152	.246	.346	.448						
15	.010	.062	.136	.220	.311	.405	.500					
16	.010	.055	.122	.199	.282	.368	.456					
17	.008	.050	.111	.181	.257	.337	.418	.500				
18	.007	.045	.101	.166	.236	.310	.385	.462				
19	.006	.041	.092	.153	.218	.286	.356	.428	.500			
20	.006	.037	.085	.141	.202	.266	.331	.398	.466			
21	.005	.034	.079	.131	.188	.247	.309	.372	.436	.500		
22	.005	.032	.073	.122	.175	.231	.289	.349	.409	.470		
23	.004	.030	.068	.114	.164	.217	.272	.328	.385	.442	.500	
24	.004	.027	.064	.107	.154	.204	.256	.309	.363	.418	.473	
25	.004	.025	.060	.101	.145	.193	.242	.292	.343	.395	.448	.500

For $k > \frac{n}{2}$:

$$\hat{p}_{k,n} = 1 - \hat{p}_{n+1-k,n} .$$

This table was computed from [4] by linear interpolation, except for the case $k = 2$ for which (3.2.5.2) was solved directly and the solution rounded off to three places.

For large values of n, and np, $n(1-p)$ both not small, the central limit theorem makes it possible to replace (3.2.5.1) by the approximate equality

$$h(p;k,n) \approx 1 - \Phi(u)$$

where $\Phi(u)$ is the area under the normal probability curve from $-\infty$ to u and

$$u = (k-np) / \sqrt{np(1-p)} .$$

3.3. Theory of S-shaped functions.

3.3.1. Definitions.

3.3.1.1. We shall say that the function $f(p)$ belongs to class Ψ when it satisfies the following conditions

(i) $f(p)$ is continuous and $0 \leq f(p) \leq 1$ for $0 \leq p \leq 1$ and $f(p)$ is differentiable for $0 < p < 1$.

(ii) the function

$$\sigma_f(p) = \frac{f(p)}{p} \cdot \frac{1-p}{1-f(p)}$$

is non-decreasing.

3.3.1.2. We say that $f(p)$ belongs to class Ψ when it satisfies (i) and

(ii') $\sigma_f(p)$ is strictly increasing,

(iii) $f(0) = 0$, $f(1) = 1$.

3.3.1.3. We say that $f(p)$ belongs to class Ψ_0 when it belongs to Ψ and satisfies the condition

(iv) $\sigma_f(p)$ assumes the value 1 in $0 < p < 1$.

3.3.2. Remarks.

From the definition of σ_f follows

$$(3.3.2.1) \qquad f = \frac{p\sigma_f}{1-p+p\sigma_f} = \frac{1}{1 + \frac{1-p}{p\sigma_f}} ,$$

hence a function $f \varepsilon \overline{\Psi}$ is strictly increasing for every p such that $0 < \sigma_f(p) < \infty$.

Clearly $\sigma_f(p) = 1$ for exactly those p for which $f(p) = p$. Since for $f \varepsilon \Psi$ the function $\sigma_f(p)$ assumes every value at most once, it follows that Ψ_0 consists of exactly those $f \varepsilon \Psi$ for which the equation $f(p) = p$ has a unique root in $0 < p < 1$.

Let $f \varepsilon \Psi_0$ and $f(p^*) = p^*$, $0 < p^* < 1$, hence $\sigma_f(p) < 1$ for $p < p^*$, $\sigma_f(p) > 1$ for $p > p^*$. From (3.3.2.1) we see that $f(p) < p$ for $p < p^*$, $f(p) > p$ for $p > p^*$. The functions of class Ψ_0 are therefore S-shaped in the sense of 3.2.4.

Let now $f \varepsilon \overline{\Psi}$. If $f(0) > 0$ then $\sigma_f(0) = +\infty$, hence $\sigma_f(p) = +\infty$ for $0 \leq p \leq 1$ and according to (3.3.2.1) $f(p) \equiv 1$ for $0 \leq p \leq 1$. If $f(1) < 1$ then $\sigma_f(1) = 0$, hence $\sigma_f(p) = 0$

and $f(p) = 0$ for $0 \leq p \leq 1$. A function $f \in \overline{\Psi}$ therefore is either identically 0, or identically 1, or it maps the closed interval $[0,1]$ onto the closed interval $[0,1]$. This mapping need not be one-to-one, as shown by the example

$$f(p) = \begin{cases} 0 & \text{for } 0 \leq p \leq \frac{1}{2} \\ 4(p-\frac{1}{2})^2 & \text{for } \frac{1}{2} \leq p \leq 1 \end{cases} .$$

3.3.3. Theorem.

Let $f, g \in \overline{\Psi}$ and at least one of f, g be neither identically 0 nor identically 1. Furthermore, let either

a) $f(p) \geq g(p)$ for $0 < p < 1$ and at least one of f, g be in Ψ

or

b) $f(p) > g(p)$ for $0 < p < 1$.

Then the function

$$(3.3.3.1) \quad h(p) = pf(p) + (1-p)g(p)$$

is in Ψ.

Proof: one verifies directly that h satisfies (i) and (iii). To prove (ii'), that is $\sigma_h'(p) > 0$ for $0 < p < 1$, we shall prove the equivalent statement

$$(3.3.3.2) \quad p(1-p)\,h' > h(1-h) \qquad \text{for } 0 < p < 1 .$$

From (3.3.3.1) follows

$$p(1-p)h' = pp(1-p)f' + (f-g)p(1-p) + (1-p)p(1-p)g'$$

$$h(1-h) = pf(1-f) + p(1-p)(f-g)^2 + (1-p)g(1-g) .$$

Since $f, g \in \overline{\Psi}$ we have $p(1-p)f' \geq f(1-f)$, $p(1-p)g' \geq g(1-g)$, and in case a) at least one of these inequalities is strict, and we have $0 \leq f - g \leq 1$ hence $f-g \geq (f-g)^2$. In case b) we have $f-g > (f-g)^2$ for $0 < p < 1$, so that (3.3.3.2) holds in either case.

3.3.4. Theorem.

If $f, g \in \Psi$, then $h(p) = f[g(p)] \in \Psi$.

Proof: properties (i) and (iii) for h are obvious, and (ii') follows from

$$\sigma_h(p) = \sigma_f[g(p)] \cdot \sigma_g(p) .$$

3.4. Mean path and mean cut.

3.4.1. Let $\underline{X} = (X_1, X_2, \ldots, X_n)$ and $\phi(\underline{X}) = \phi(X_1, \ldots, X_n)$ have the same meaning as in 3.1.2, and let

$$S = S(\underline{X}) = X_1 + X_2 + \ldots + X_n$$

be the size of the random vector $\underline{X}$, i.e. the number of components which perform. Then S is a random variable with the binomial distribution, ϕ a random variable with the probability distribution $\Pr\{\phi = 1\} = h(p)$, $\Pr\{\phi = 0\} = 1-h(p)$, but S and ϕ are dependent random variables. It is natural to consider the conditional expectations

$$E(S \mid \phi = 1) = L(p)$$

$$E(n - S \mid \phi = 0) = W(p)$$

and to call $L(p)$ the <u>mean path</u>, $W(p)$ the <u>mean cut</u> for the structure ϕ.

3.4.2. Theorem.

In order that the reliability function h of a structure of order n is strictly increasing for $0 < p < 1$ it is necessary and sufficient that

$$(3.4.2.1) \qquad L(p) + W(p) > n \quad .$$

This condition is equivalent with

$$(3.4.2.2) \qquad \operatorname{cov}(S, \phi) > 0 \quad .$$

Proof: we shall prove somewhat more specifically that for any fixed p, $0 < p < 1$, each of the conditions (3.4.2.1), (3.4.2.2) is necessary and sufficient to have $h'_\phi(p) > 0$.

Writing h in short for h_ϕ we have from (3.2.1)

$$h'(p) = \frac{1}{p(1-p)} \left[\sum_{j=0}^{n} j\, A_j p^j (1-p)^{n-j} - np \sum_{j=0}^{n} A_j p^j (1-p)^{n-j} \right] =$$

$$(3.4.2.3) \qquad = \frac{1}{p(1-p)} \left[E(S\,\phi) - E(S)E(\phi) \right] =$$

$$= \frac{1}{p(1-p)} \operatorname{cov}(S, \phi)$$

which shows that (3.4.2.2) is equivalent with $h'(p) > 0$.

Since

$$E(S|\phi = 1) = \sum_{j=0}^{n} j \frac{\Pr(S=j, \phi=1)}{\Pr(\phi = 1)} =$$

$$= \frac{1}{E(\phi)} \sum_{j=0}^{n} j\, A_j p^j (1-p)^{n-j} = \frac{E(\phi S)}{E(\phi)}$$

and

$$E(S|\phi = 0) = \sum_{j=0}^{n} j \frac{\Pr(S=j, \phi=0)}{\Pr(\phi = 0)} =$$

$$= \frac{1}{1-E(\phi)} \sum_{j=0}^{n} j\, A_j^* p^j (1-p)^{n-j} = \frac{E(S)-E(\phi S)}{1-E(\phi)} ,$$

we obtain

$$L(p) + W(p) = \frac{E(\phi S)}{E(\phi)} + n - \frac{E(S)-E(\phi S)}{1 - E(\phi)} =$$

(3.4.2.4)

$$= n + \frac{\mathrm{cov}(\phi, S)}{h(p)[1-h(p)]}$$

so that (3.4.2.1) and (3.4.2.2) are equivalent.

3.4.3. Theorem.

Let ϕ be a structure of order n and $h(p)$ its reliability function. In order that $\sigma_h(p)$ is strictly increasing it is necessary and sufficient that

(3.4.3.1) $L(p) + W(p) > n + 1$

and this condition is equivalent with

(3.4.3.2) $\mathrm{cov}(S - \phi, \phi) > 0$.

Proof: we have seen in 3.3.3. that $\sigma_h'(p) > 0$ at a point p, $0 < p < 1$, is equivalent with inequality (3.3.3.2). From (3.4.2.3) follows

$$p(1-p)h'(p) = E(S\phi) - E(S)E(\phi) = \mathrm{cov}(S, \phi)$$

and since

$$h(p)[1 - h(p)] = \mathrm{var}(\phi)$$

inequality (3.3.3.2) is equivalent with

$$\mathrm{cov}(S - \phi, \phi) = \mathrm{cov}(S, \phi) - \mathrm{var}(\phi) > 0 \ .$$

From (3.4.2.4) follows

$$L(p) + W(p) = n + 1 + \frac{\mathrm{cov}(S - \phi, \phi)}{h(p)[1-h(p)]}$$

which shows that (3.4.3.1) and (3.4.3.2) are equivalent.

3.5. Structures coherent in probability.

3.5.1. Let ϕ and S again have the same meaning as in 3.4. We shall say that a structure ϕ of order n is <u>coherent in probability</u> when

(a) $\Pr\{\phi = 1 \mid S = k\} \leq \Pr\{\phi = 1 \mid S = k+1\}$ for $k=0,1,\ldots,n-1$,

(b) $\Pr\{\phi = 1 \mid S = 0\} = 0$, $\Pr\{\phi = 1 \mid S = n\} = 1$.

3.5.2. Theorem.

A structure is coherent in probability if and only if its path-numbers satisfy the inequalities (2.7.4.2) and (2.7.4.1), and $A_0 = 0$, $A_1 = 1$.

Proof: since $\Pr\{\phi = 1, S = k\} = A_k p^k (1-p)^{n-k}$ and $\Pr\{S = k\} = \binom{n}{k} p^k (1-p)^{n-k}$, we have

$$\Pr\{\phi = 1 \mid S = k\} = A_k / \binom{n}{k}, \quad k = 0,1,\ldots,n$$

for any structure, and our theorem follows immediately.

3.5.3. Corollary.

If a structure is coherent, then it is coherent in probability.

This follows from 3.5.2 and 2.7.4. The converse statement is not true, as shown by the example of 2.7.3.1. The structure of order 3 described there satisfies (2.7.4.2) hence is coherent in probability, but is not even semi-coherent.

3.5.4. Theorem.

If ϕ is coherent in probability then its reliability function $h(p)$ is non-decreasing for $0 \leq p \leq 1$.

Proof: from (3.2.1) we have

(3.5.4.1) $h'(p) = \sum_{j=0}^{n-1} [(j+1) A_{j+1} - (n-j)A_j]p^j(1-p)^{n-j-1}$

and by 3.5.2 and (2.7.4.1) we obtain $h'(p) \geq 0$.

3.6. Reliability functions of semi-coherent and coherent structures.

3.6.1. Theorem.

If ϕ is a semi-coherent structure then either $h_\phi \equiv 0$, or $h_\phi \equiv 1$, or $h_\phi \equiv p$, or $h_\phi \in \Psi$.

Proof: to use induction on the order of ϕ , we first consider $n=1$. The three possible structures $\beta_0, \beta_1, \beta_2$ of order 1 listed in 2.7.3 have the reliability functions $0, p, 1,$ respectively. We now assume our statement to be true for order n , and consider a semi-coherent structure ϕ of order $n+1$. According to 2.7.2 we have

(3.6.1.1) $$\phi(X_1,\ldots,X_n,X_{n+1}) = X_{n+1}\lambda(X_1,\ldots,X_n) + (1-X_{n+1})\mu(X_1,\ldots,X_n)$$

with λ and μ semi-coherent of order n and $\lambda(X_1,\ldots,X_n) \geq \mu(X_1,\ldots,X_n)$ for all $(X_1,\ldots,X_n)$. From (3.2.2) we have

$$h_\phi(p) = p\, h_\lambda(p) + (1-p)\, h_\mu(p) \quad .$$

If $\lambda(X_1,\ldots,X_n) = \mu(X_1,\ldots,X_n)$ for all $(X_1,\ldots,X_n)$ then $h_\phi(p) = h_\lambda(p)$ for $0 \leq p \leq 1$ and our statement is true by the assumption of induction. If $\lambda(X_1,\ldots,X_n) > \mu(X_1,\ldots,X_n)$ for some value of $(X_1,\ldots,X_n)$ then $h_\lambda(p) > h_\mu(p)$ for $0 < p < 1$ and $h_\phi \in \Psi$ according to 3.3.3.

3.6.2. Corollary.

If ϕ is a coherent structure then either

$$h_{\phi}(p) = p \quad \text{or} \quad h_{\phi} \in \Psi \quad .$$

This follows immediately from 3.6.1 by observing that for ϕ coherent $h(0) = 0$ and $h(1) = 1$ hence h is neither identically 0 nor identically 1.

3.6.3. Theorem.

If ϕ is a coherent structure, then $h_{\phi} \in \Psi_{o}$ if and only if $A_1 = 0$ and $A_{n-1} = n$.

Proof: since for ϕ coherent $A_o = 0$, $A_n = 1$ we have

$$\sigma_h(p) = \frac{\sum_{j=1}^{n} A_j p^j (1-p)^{n-j}}{p} \cdot \frac{1-p}{\sum_{i=0}^{n-1} [\binom{n}{i} - A_i] p^i (1-p)^{n-i}} =$$

$$= \frac{\sum_{j=1}^{n} A_j p^{j-1} (1-p)^{n-j}}{\sum_{i=0}^{n-1} [\binom{n}{i} - A_i] p^i (1-p)^{n-1-i}}$$

and

$$\lim_{p \to 0} \sigma_h(p) = \frac{A_1}{1-A_o} = A_1$$

(3.6.3.1)

$$\lim_{p \to 1} \sigma_h(p) = \frac{A_n}{\binom{n}{n-1} - A_{n-1}} = \frac{1}{n - A_{n-1}} \quad .$$

For $h_{\phi} \in \Psi$ the function $\sigma_h(p)$ is strictly increasing and it assumes the value 1 in $0 < p < 1$ if and only if $\lim_{p \to 0} \sigma_h(p) < 1$ and $\lim_{p \to 1} \sigma_h(p) > 1$ which together with (3.6.3.1)

shows that $A_1 = 0$, $A_{n-1} = n$.

3.6.4. Inequalities for $h(p)$ in terms of length and width.

Let ℓ and w denote the length and the width of a structure ϕ, as defined in 2.5. Then $A_i = 0$ for $i = 0, 1, \ldots, \ell - 1$, and $A_i^* = 0$ for $i = n-w+1, \ldots, n$, so that the reliability function h of ϕ can be written

$$(3.6.4.1) \quad h(p) = \sum_{i=\ell}^{n} A_i p^i (1-p)^{n-i} = 1 - \sum_{i=0}^{n-w} A_i^* p^i (1-p)^{n-i} \quad .$$

This immediately yields the inequalities

$$(3.6.4.2) \quad A_\ell p^\ell (1-p)^{n-\ell} \leq h(p) \leq 1 - A_{n-w}^* p^{n-w} (1-p)^w$$

which show that for $p \to 0$ the function $h(p)$ tends to 0 not faster than p^ℓ, that is not faster than in the case of components in series; and for $p \to 1$ it tends to 1 not faster than $1 - (1-p)^w$, that is not faster than in the case of w parallel components.

Inequalities (3.6.4.1) hold without any assumptions on ϕ. If ϕ is coherent, we have inequalities (2.7.4.2) from which one obtains

$$A_i \geq \frac{\binom{n}{i}}{\binom{n}{\ell}} A_\ell \qquad \text{for } i = \ell, \ell+1, \ldots, n$$

and

$$A_i^* \geq \frac{\binom{n}{i}}{\binom{n}{w}} A_{n-w}^* \qquad \text{for } i = 0, 1, \ldots, n-w \quad .$$

From these inequalities and (3.6.4.1) follow the bounds for $h(p)$

$$(3.6.4.3) \quad \frac{A_\ell}{\binom{n}{\ell}} \sum_{i=\ell}^{n} \binom{n}{i} p^i (1-p)^{n-i} \leq h(p) \leq 1 - \frac{A_{n-w}^*}{\binom{n}{w}} \sum_{i=0}^{n-w} \binom{n}{i} p^i (1-p)^{n-i} \quad .$$

shows that $A_1 = 0$, $A_{n-1} = n$.

3.6.4. Inequalities for $h(p)$ in terms of length and width.

Let ℓ and w denote the length and the width of a structure ϕ, as defined in 2.5. Then $A_i = 0$ for $i = 0, 1, \ldots, \ell - 1$, and $A_i^* = 0$ for $i = n-w+1, \ldots, n$, so that the reliability function h of ϕ can be written

$$(3.6.4.1)\quad h(p) = \sum_{i=\ell}^{n} A_i p^i (1-p)^{n-i} = 1 - \sum_{i=0}^{n-w} A_i^* p^i (1-p)^{n-i} .$$

This immediately yields the inequalities

$$(3.6.4.2)\quad A_\ell p^\ell (1-p)^{n-\ell} \leq h(p) \leq 1 - A_{n-w}^* p^{n-w} (1-p)^w$$

which show that for $p \to 0$ the function $h(p)$ tends to 0 not faster than p^ℓ, that is not faster than in the case of components in series; and for $p \to 1$ it tends to 1 not faster than $1 - (1-p)^w$, that is not faster than in the case of w parallel components.

Inequalities (3.6.4.1) hold without any assumptions on ϕ. If ϕ is coherent, we have inequalities (2.7.4.2) from which one obtains

$$A_i \geq \frac{\binom{n}{i}}{\binom{n}{\ell}} A_\ell \qquad \text{for } i = \ell, \ell+1, \ldots, n$$

and

$$A_i^* \geq \frac{\binom{n}{i}}{\binom{n}{w}} A_{n-w}^* \qquad \text{for } i = 0, 1, \ldots, n-w .$$

From these inequalities and (3.6.4.1) follow the bounds for $h(p)$

$$(3.6.4.3)\quad \frac{A_\ell}{\binom{n}{\ell}} \sum_{i=\ell}^{n} \binom{n}{i} p^i (1-p)^{n-i} \leq h(p) \leq 1 - \frac{A_{n-w}^*}{\binom{n}{w}} \sum_{i=0}^{n-w} \binom{n}{i} p^i (1-p)^{n-i} .$$

3.6.5. Qualitative remarks.

In 3.2.4 we described some properties of reliability functions which appeared desirable for practical structures. The preceding theorems tell us what kind of structures have these properties: If ϕ is coherent then h_ϕ has properties a) and b) of 3.2.4 (according to 3.6.2). Furthermore, if ϕ is coherent then the conditions on path-numbers $A_1 = 0$, $A_{n-1} = n$ characterize it as having a reliability function h_ϕ of class $_o$, hence S-shaped and lending itself to the process of iterated compositions.

The class of coherent structures is very large. It contains among others the two-terminal networks whose reliability has been studied by Moore and Shannon in their fundamental paper [1] and all k out of n structures.

REFERENCES

[1] E. F. Moore and C. E. Shannon, "Reliable circuits using less reliable relays", Journ. of the Franklin Institute, v. 262 (1956), pp. 191-208.

[2] J. von Neumann, "Probabilistic logics and the synthesis of reliable organisms from unreliable components", Automata Studies, pp. 43-98, Annals of Math. Studies No. 34, Princeton University Press 1956.

[3] Hisashi Mine, "Reliability of physical system". I. R. E. Transactions, P.G.I.T., v. IT-5, Special Supplement May 1959, pp. 138-150.

[4] Tables of the binomial probability distribution, National Bureau of Standards, Applied Mathematics Series 6, 1950.